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Complexifications and real forms of Hamiltonian structures

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Abstract. We consider generalizations of the standard Hamiltonian dynamics to complex dynamical variables and introduce the notions of real Hamiltonian form in analogy with the notion of real forms for a simple Lie algebra. Thus to each real Hamiltonian system we are able to relate several nonequivalent ones. On the example of the complex Toda chain we demonstrate how starting from a known integrable Hamiltonian system (*e.g.* the Toda chain) one can complexify it and then project onto different real forms.

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1 Introduction

Recently the so-called complex Toda chain (CTC) was shown to describe N-soliton interactions in the adiabatic approximation [1–3]. The complete integrability of the CTC is a direct consequence of the integrability of the real (standard) Toda chain (TC); it was also shown that CTC allows several dynamical regimes that are qualitatively different from the one of RTC [2]. These results as well as the clear algebraic structures lying behind the integrability of CTC (such as, *e.g.* Lax representation) were the stimulation for the present work.

We start from a standard (real) Hamiltonian system $\mathcal{H} \equiv \{\omega, H, \mathcal{M}\}$ with n degrees of freedom and Hamiltonian H depending analytically on the dynamical variables. It is known that such systems can be complexified and then written as a Hamiltonian system with $2n$ (real) degrees of freedom. Our main aim is to show that to each compatible involutive automorphism $\widetilde{\mathcal{C}}$ of the complexified phase space we can relate a real Hamiltonian form of the initial system with n degrees of freedom. Just like to each complex Lie algebra one associates several inequivalent real forms, so to each $\mathcal{H}$ we associate several inequivalent real forms $\mathcal{H}_{\mathbb{R}} \equiv \{\omega_{\mathbb{R}}, H_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}}\}$. Like the initial system $\mathcal{H}$, the real form is defined on a manifold $\mathcal{M}_{\mathbb{R}}$ with n real degrees of freedom. Provided $\widetilde{\mathcal{C}}(H) = H$ the dynamics on the real form will be well defined and will coincide with the dynamics on $\mathcal{M}_{\mathbb{C}}$ restricted to $\mathcal{M}_{\mathbb{R}}$. If the initial system $\mathcal{H}$ is integrable then its real Hamiltonian forms will also be integrable. We pay special attention to the connection with integrable systems and the possibility they offer to define a class of new integrable systems starting from an initial one.

Examples of non-standard (or "twisted") real forms have already been studied by Evans and Madsen [4] in connection with the problem of positive kinetic energy terms in the Lagrangian description and with emphasis on conformal WZNW models. Examples of indefinite-metric Toda chain (IMTC) has already been studied by Kodama and Ye [5]. In particular they note that while the solutions of the TC model are regular for all t, the solutions of the IMTC model develop singularities for finite values of t.

The approach we follow here is different and more general than the ones in [4,5].

2 Real Hamiltonian forms

The idea of reducing the dynamics on a complexified phase space to a real space and to obtain in this way a new dynamical system on a phase space which is isomorphic to the initial real space could be pursued in different routes. One could start with defining complex structure (or equivalently Kähler polarization), then find the invariant polarizations which are naturally associated with an integrable

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system and to vary among them. Polarizations are best known in the context of geometric quantization [6], see also [7,8]. Another approach is to define a "complex conjugation operator" and to reduce dynamics to the real space associated with this operator.

The approach we will follow in this paragraph is inspired by the basic idea of construction of real forms for simple Lie algebras [9]. A basic tool in the following construction is a Cartan-like involutive automorphism $\widetilde{\mathcal{C}}$ defined below, which plays the rôle of a "complex conjugation operator".

We start by introducing the involutive automorphism $\mathcal{C}$ on the phase space $\mathcal{M}^{(n)}$ and on its dual[1] by:

$$\mathcal{C}\left(\{F,G\}\right) = \{\mathcal{C}(F), \mathcal{C}(G)\}, \qquad \mathcal{C}^2 = 1\!\!1, \tag{1}$$

where F, $G \in \mathcal{F}(\mathcal{M}^{(n)})$ are real analytic functions on $\mathcal{M}^{(n)}$. The involution acts on them by:

$$\mathcal{C}(F(p_1,\ldots,p_n,q_1,\ldots,q_n)) = F((\mathcal{C}(p_1),\ldots,\mathcal{C}(p_n),\mathcal{C}(q_1),\ldots,\mathcal{C}(q_n))). \tag{2}$$

Since $\mathcal{C}$ has eigenvalues 1 and -1 it naturally splits $\mathcal{M}^{(n)}$ into two subspaces $\mathcal{M}^{(n)} = \mathcal{M}_+^{(n)} \oplus \mathcal{M}_-^{(n)}$ such that

$$\begin{aligned} \mathcal{C}\mathbf{X} &= \mathbf{X} \qquad \text{for} \quad \mathbf{X} \in \mathcal{M}_+^{(n_+)}, \\ \mathcal{C}\mathbf{Y} &= -\mathbf{Y} \qquad \text{for} \quad \mathbf{Y} \in \mathcal{M}_-^{(n_-)}. \end{aligned} \tag{3}$$

where $n_\pm = \dim \mathcal{M}_\pm^{(n_\pm)}$. We will assume also that the starting Hamiltonian H is invariant with respect to $\mathcal{C}$:

$$\mathcal{C}(H) = H. \tag{4}$$

In terms of vector fields $X, Y \in T\mathcal{M}$ and the lifted involution $T\mathcal{C} : T\mathcal{M} \to T\mathcal{M}$ we have:

$$\omega(T\mathcal{C}(X), T\mathcal{C}(Y)) = T\mathcal{C}(\omega(X,Y)), \quad (T\mathcal{C})^2 = 1\!\!1. \tag{5}$$

ω being the symplectic structure associated with the Poisson brackets in (1).

The complexified phase space $\mathcal{M}_{\mathbb{C}}^{(2n)}$ can be viewed as a the linear space $\mathcal{M}^{(n)}$ over the field of complex numbers:

$$\mathcal{M}_{\mathbb{C}}^{(2n)} = \mathcal{M}^{(n)} \oplus \mathrm{i}\mathcal{M}^{(n)}.$$

In other words the dynamical variables p_k, q_k in $\mathcal{M}_{\mathbb{C}}^{(n)}$ now may take complex values. The real analytic functions F and G as well as the automorphism $\mathcal{C}$ can naturally be extended to $\mathcal{M}_{\mathbb{C}}^{(2n)}$. In $\mathcal{M}_{\mathbb{C}}^{(n)}$ along with $\mathcal{C}$ we can introduce also the complex conjugation $*$. In this construction obviously $\mathcal{C}$ commutes with $*$ and their composition $\widetilde{\mathcal{C}} = \mathcal{C} \circ * = * \circ \mathcal{C}$ is an involutive automorphism on $\mathcal{M}_{\mathbb{C}}^{(2n)}$.

[1] In general we should have different notations of $\mathcal{C}$ in these spaces [10]. However, since our phase space is a vector space with some abuse of notations we will use the same letter for both realizations.

The real form $\mathcal{M}_{\mathbb{R}}^{(n)}$ of the phase space is the subspace of $\mathcal{M}_{\mathbb{C}}^{(2n)}$ invariant with respect to $\widetilde{\mathcal{C}}$:

$$\mathcal{M}_{\mathbb{R}}^{(n)} = \mathcal{M}_+^{(n_+)} \oplus \mathrm{i}\mathcal{M}_-^{(n_-)}. \tag{6}$$

Indeed any element of $\mathcal{M}_{\mathbb{R}}^{(n)}$ can be represented as:

$$\mathbf{Z} = \mathbf{X} + \mathrm{i}\mathbf{Y} \in \mathcal{M}_{\mathbb{R}}^{(n)}, \tag{7}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are real-valued elements of $\mathcal{M}_+^{(n_+)}$ and $\mathcal{M}_-^{(n_-)}$ respectively. The reality condition means that

$$\tilde{\mathcal{C}}\mathbf{Z} \equiv \mathcal{C}(\mathbf{Z}^*) = \mathcal{C}(\mathbf{X} - \mathrm{i}\mathbf{Y}) = \mathbf{X} + \mathrm{i}\mathbf{Y} = \mathbf{Z}, \tag{8}$$

where we have made use of equation (3).

Equation (1) guarantees that each of the subspaces $\mathcal{M}_+^{(n_+)}$, $\mathcal{M}_-^{(n_-)}$ and $\mathcal{M}_{\mathbb{R}}^{(n)}$ is a symplectic subspace of $\mathcal{M}_{\mathbb{C}}^{(2n)}$. Indeed, let us assume that the symplectic structure on $\mathcal{M}^{(n)}$ is defined by the canonical 2-form

$$\omega = \sum_{k=1}^{n} \mathrm{d}p_k \wedge \mathrm{d}q_k.$$

Then on $\mathcal{M}_+^{(n_+)}$ and $\mathcal{M}_-^{(n_-)}$ we have

$$\omega_+ = \sum_{k=1}^{n_+} \mathrm{d}p_k^+ \wedge \mathrm{d}q_k^+. \qquad \omega_- = \sum_{k=1}^{n_-} \mathrm{d}p_k^- \wedge \mathrm{d}q_k^-,$$

where p_k^+, q_k^+ (resp. p_k^-, q_k^-) are linearly independent non-vanishing basic elements in $\mathcal{M}_+^{(n_+)}$ (resp. $\mathcal{M}_-^{(n_-)}$). With respect to the automorphism $\mathcal{C}$ they satisfy:

$$\mathcal{C}(p_k^\pm) = \pm p_k^\pm, \qquad \mathcal{C}(q_k^\pm) = \pm q_k^\pm, \tag{9}$$

for all $k = 1, \ldots, n_\pm$.

The complexification of the dynamical variables F and G means that they are now analytic functions of the complex arguments $q_k = q_{k,0} + \mathrm{i}q_{k,1}$, $p_k = p_{k,0} + \mathrm{i}p_{k,1}$, $k = 1, \ldots, n$, or equivalently, of

$$p_k^\pm = p_{k,0}^\pm + \mathrm{i}p_{k,1}^\pm, \qquad q_k^\pm = q_{k,0}^\pm + \mathrm{i}q_{k,1}^\pm, \quad k = 1, \ldots, n_\pm. \tag{10}$$

We will say that $F, G \in \mathcal{F}(\mathcal{M}_{\mathbb{R}}^{(n)})$ if their arguments are restricted to $\mathcal{M}_{\mathbb{R}}$. Then F and G will satisfy the analog of equation (2) with $\mathcal{C}$ replaced by $\widetilde{\mathcal{C}}$. Due to equation (1) their Poisson bracket $\{F,G\} \in \mathcal{F}(\mathcal{M}_{\mathbb{R}}^{(n)})$ too. If we choose the Hamiltonian $H \in \mathcal{F}(\mathcal{M}_{\mathbb{R}}^{(n)})$ then

$$\widetilde{\mathcal{C}}(\mathrm{d}H) = \mathrm{d}H. \tag{11}$$

The evolution of the dynamical variable F:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{H, F\}, \tag{12}$$

becomes naturally restricted to $\mathcal{M}_{\mathbb{R}}^{(n)}$. Rewriting (12) into its equivalent form:

$$\omega(X_H, \cdot) = \mathrm{d}H, \tag{13}$$

and making use of (5) we see that the vector field X_H must also satisfy

$$\widetilde{C}(X_H) = X_{\widetilde{C}(H)} = X_H \tag{14}$$

on $\mathcal{M}_{\mathbb{R}}^{(n)}$. The symplectic form restricted on $\mathcal{M}_{\mathbb{R}}^{(n)}$ also becomes real and equals

$$\omega_{\mathbb{R}} = \sum_{k=1}^{n_+} \mathrm{d}p_{k,0}^+ \wedge \mathrm{d}q_{k,0}^+ - \sum_{k=1}^{n_-} \mathrm{d}p_{k,1}^- \wedge \mathrm{d}q_{k,1}^-, \tag{15}$$

where $p_{k,0}^+, q_{k,0}^+, k = 1, \ldots, n_+$ and $p_{k,1}^-, q_{k,1}^-, k = 1, \ldots, n_-$ are the basic elements in $\mathcal{M}_{\mathbb{R}}^{(n)}$.

If $\mathcal{M}_{\mathbb{C}}^{(2n)}$ is endowed with Hamiltonian which is "real" with respect to $\widetilde{\mathcal{C}}$ and whose vector field X_H satisfies (14) then the restriction of the dynamics on $\{\mathcal{M}_{\mathbb{R}}^{(n)}, \omega_{\mathbb{R}}, H_{\mathbb{R}}\}$ will be well defined and will coincide with the dynamics on $(\mathcal{M}_{\mathbb{C}}^{(2n)}, \omega, H)$ restricted to $\mathcal{M}_{\mathbb{R}}^{(n)}$.

Thus we obtain a well defined dynamical system with n degrees of freedom $\mathcal{H}_{\mathbb{R}} \equiv \{\mathcal{M}_{\mathbb{R}}, \omega_{\mathbb{R}}, H\}$ which we call a real form of the initial Hamiltonian system $\mathcal{H} \equiv \{\mathcal{M}, \omega, H\}$.

Let us outline our construction on the simplest possible case when the initial Hamiltonian system is completely integrable and is parametrized by its action-angle variables $\{I_k^\pm, \phi_k^\pm\}$, $k = 1, \ldots, n_\pm$. Let us also assume that the involution $\mathcal{C}$ is of the form $\mathcal{C}(\mathrm{d}I_k^\pm) = \pm\mathrm{d}I_k^\pm$, $\mathcal{C}(\mathrm{d}\phi_k^\pm) = \pm\mathrm{d}\phi_k^\pm$ for $k = 1, \ldots n_\pm$, $n_+ + n_- = n$. The complexification renders all $I_k^\pm$ and $\phi_k^\pm$ complex: $I_k^{\mathbb{C},\pm} = I_{k,0}^\pm + \mathrm{i}I_{k,1}^\pm$ and $\phi_k^{\mathbb{C},\pm} = \phi_{k,0}^\pm + \mathrm{i}\phi_{k,1}^\pm$ and the automorphism $\widetilde{\mathcal{C}}$ has the form:

$$\widetilde{\mathcal{C}}\left(\mathrm{d}I_k^{\mathbb{C},\pm}\right) = \pm\left(\mathrm{d}I_k^{\mathbb{C},\pm}\right)^*, \qquad \widetilde{\mathcal{C}}\left(\mathrm{d}\phi_k^{\mathbb{C},\pm}\right) = \pm\left(\mathrm{d}\phi_k^{\mathbb{C},\pm}\right)^*. \tag{16}$$

The automorphism $\widetilde{\mathcal{C}}$ obviously satisfies the condition (14). In order to satisfy also (4) we need to assume that H is an even function of all $I_k^- \in \mathcal{M}_-$. Then the restriction on $\mathcal{M}_{\mathbb{R}}$ according to (7) and (16) means restricting all $I_k^+, \phi_k^+ \in \mathcal{M}_+$ to be real, while all $I_k^-, \phi_k^- \in \mathcal{M}_-$ become purely imaginary. Then:

$$H_{\mathbb{R}} = H\left(I_{1,0}^+, \ldots, I_{n_+,0}^+, \mathrm{i}I_{1,1}^-, \ldots, \mathrm{i}I_{n_-,1}^-\right),$$
$$\omega_{\mathbb{R}} = \sum_{k=1}^{n_+} \mathrm{d}I_{k,0}^+ \wedge \mathrm{d}\phi_{k,0}^+ - \sum_{k=1}^{n_-} \mathrm{d}I_{k,1}^- \wedge \mathrm{d}\phi_{k,1}^-, \tag{17}$$

and obviously we have again a completely integrable Hamiltonian system.

To those who prefer to work only with real-valued dynamical variables we propose an alternative way of deriving the real Hamiltonian forms. This will also make clear why we insisted on the Hamiltonian H and the dynamical variables F to be analytic functions of q_k and p_k. This condition means that $H = H_0 + \mathrm{i}H_1$ must satisfy:

$$\frac{\partial H_0}{\partial q_{k,0}} = \frac{\partial H_1}{\partial q_{k,1}}, \qquad \frac{\partial H_1}{\partial q_{k,0}} = -\frac{\partial H_0}{\partial q_{k,1}} \cdot \tag{18}$$

Then the equations of motion (12) with $F = \mathbf{z}$ and $F = \tilde{\mathbf{z}}$ become:

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \begin{pmatrix} -S_0^+ & 0 \\ 0 & S_0^- \end{pmatrix} \nabla_{\mathbf{z}} H_0, \qquad \frac{\mathrm{d}\tilde{\boldsymbol{z}}}{\mathrm{d}t} = \begin{pmatrix} -S_0^- & 0 \\ 0 & S_0^+ \end{pmatrix} \nabla_{\tilde{\mathbf{z}}} H_0, \tag{19}$$

with $\mathbf{z} = \begin{pmatrix} \mathbf{x}_+ \\ \mathbf{y}_- \end{pmatrix}$, $\tilde{\mathbf{z}} = \begin{pmatrix} \mathbf{x}_- \\ \mathbf{y}_+ \end{pmatrix}$, $\mathbf{x}_\pm = \begin{pmatrix} \boldsymbol{p}_\pm^0 \\ \boldsymbol{q}_\pm^0 \end{pmatrix}$, $\mathbf{y}_\pm = \begin{pmatrix} \boldsymbol{p}_\pm^1 \\ \boldsymbol{q}_\pm^1 \end{pmatrix}$ and $S_0^\pm = \begin{pmatrix} 0 & \mathbb{1}_{n_\pm} \\ -\mathbb{1}_{n_\pm} & 0 \end{pmatrix}$, $(\nabla_{\mathbf{z}} H_0)_k = \partial H_0/\partial z_k$. Note that equation (19) has the form of a standard Hamiltonian equation of motion for a system of $2n$ degrees of freedom with Hamiltonian H_0. The phase space of this system is spanned by the $2n$-component real-valued vectors $\mathbf{z}$ and $\tilde{\boldsymbol{z}}$ and the symplectic structure introduced by $S_0^\pm$ corresponds to the following 2-form:

$$\omega^{(2n)} = \sum_{\epsilon=\pm} \sum_{k=1}^{n} \left(\mathrm{d}p_{k,0}^\epsilon \wedge \mathrm{d}q_{k,0}^\epsilon - \mathrm{d}p_{k,1}^\epsilon \wedge \mathrm{d}q_{k,1}^\epsilon\right). \tag{20}$$

The involutive automorphism $\widetilde{C}$ acts on $\mathbf{z}$ and $\tilde{\mathbf{z}}$ by:

$$\widetilde{C}(\mathbf{z}) = \mathbf{z}, \qquad \widetilde{C}(\tilde{\mathbf{z}}) = -\tilde{\mathbf{z}}, \tag{21}$$

i.e., the phase space $\mathcal{M}_{\mathbb{R}}$ of the real Hamiltonian form again is the invariant subspace of $\widetilde{C}$.

The restriction of the equations (19) on $\mathcal{M}_{\mathbb{R}}$ means that $\mathrm{d}\tilde{\mathbf{z}}/\mathrm{d}t = 0$ identically. To be consistent we have to show that $\nabla_{\tilde{\mathbf{z}}_{\mathbb{R}}} H_0$ vanishes for $\tilde{\mathbf{z}} = 0$. Indeed, H_0 is the real part of a real analytic function satisfying equation (4). This means that it is an even function of $\mathbf{x}_-$ and $\mathbf{y}_+$ and therefore, its derivatives evaluated for $\mathbf{x}_- = 0$ and $\mathbf{y}_+ = 0$ vanish. Thus we have proved that the equations (19) can be consistently constrained to $\mathcal{M}_{\mathbb{R}}$.

3 Completely integrable systems

Here we consider a completely integrable Hamiltonian system whose phase space is parametrized by its action-angle variables $\{I_k^\pm, \phi_k^\pm\}$, $k = 1, \ldots, n_\pm$. We define the involutive automorphism $\mathcal{C}$ by $\mathcal{C}(\mathrm{d}I_k^\pm) = \pm\mathrm{d}I_k^\pm$, $\mathcal{C}(\mathrm{d}\phi_k^\pm) = \pm\mathrm{d}\phi_k^\pm$ and assume that the Hamiltonian is separable, *i.e.* $H = \sum_{k=1}^{n_+} h_k^+(I_k^+) + \sum_{k=1}^{n_-} h_k^-(I_k^-)$. Obviously the condition (4) requires that $h_k^-(I_k^-)$ must be even functions of I_k^-.

In purely algebraic language integrability can be characterized by the existence of a maximal rank Abelian subalgebra in the commutant of the Hamiltonian. Another important approach to completely integrable systems is based on the notion of the recursion operator – a (1,1) tensor field with vanishing Nijenhuis torsion [11]. This tensor field reflects the possibility to introduce a second symplectic structure ω_1 on $\mathcal{M}$ through:

$$\omega_1(X,Y) = \frac{1}{2}\left(\omega(TX,Y) + \omega(X,TY)\right). \qquad (22)$$

In terms of $I_k^{\pm}$, $\phi_k^{\pm}$ the tensor field T and ω_1 can be expressed by:

$$T = \sum_{k=1}^{n_+} T_k^+ + \sum_{k=1}^{n_-} T_k^-, \qquad \omega_1 = \sum_{k=1}^{n_+} \omega_{1,k}^+ + \sum_{k=1}^{n_-} \omega_{1,k}^-,$$

$$T_k^{\pm} = \lambda_k^{\pm}(I_k^{\pm})\left(\mathrm{d}I_k^{\pm} \otimes \frac{\partial}{\partial I_k^{\pm}} + \mathrm{d}\phi_k^{\pm} \otimes \frac{\partial}{\partial \phi_k^{\pm}}\right), \qquad (23)$$

$$\omega_{1,k}^{\pm} = \lambda_k^{\pm}(I_k^{\pm})\mathrm{d}I_k^{\pm} \wedge \mathrm{d}\phi_k^{\pm}, \qquad (24)$$

where $\lambda_k^{\pm}(I_k^{\pm})$ are some functions of $I_k^{\pm}$; we assume that they are real analytic functions of their variables.

In order that ω_1 also satisfy equation (5) it is enough that $\mathcal{C}(T) = T$. In particular this means that λ_k^- must be even functions of I_k^-. If this is so then we can repeat our construction also for ω_1; *i.e.*, we can complexify it and then restrict it onto $\mathcal{M}_{\mathbb{R}}$ with the result:

$$T_{\mathbb{R}} = \sum_{k=1}^{n_+} T_k^+ + \sum_{k=1}^{n_-} T_k^-, \quad \omega_{1,\mathbb{R}} = \sum_{k=1}^{n_+} \omega_{1,k}^+ - \sum_{k=1}^{n_-} \omega_{1,k}^- \qquad (25)$$

where in addition we have to replace $\lambda_k^-(I_k)$ by $\lambda_k^-(\mathrm{i}I_k)$.

Thus by construction, the restriction to other real forms preserves the Nijenhuis property of T. Due to the fact that existence of recursion operators is equivalent to integrability at least in the non-resonant case, this line of argumentation gives us another instrument to treat the integrability of real forms.

The separability of H looks rather restrictive condition. However, all integrable systems obtained by reducing a soliton equation (like, *e.g.* the nonlinear Schrödinger equation) on its N-soliton sector are separable.

4 Examples

Example 1 We illustrate our point by the paradigmical example of the Toda chain related to the $sl(n,\mathbb{C})$ algebra:

$$H_{\mathrm{TC}} = \sum_{k=1}^{n} \frac{p_k^2}{2} + \sum_{k=1}^{n-1} \mathrm{e}^{q_{k+1}-q_k}, \qquad \omega = \sum_{k=1}^{n} \mathrm{d}p_k \wedge \mathrm{d}q_k.$$

We complexify it and choose the involution as:

$$\widetilde{\mathcal{C}}(p_k) = -p_{\bar{k}}^*, \qquad \widetilde{\mathcal{C}}(q_k) = -q_{\bar{k}}^* \qquad (26)$$

where $\bar{k} = n+1-k$. As a result we obtain the following real forms of the TC model: i) for $n = 2r+1$:

$$\begin{aligned} H_{\mathrm{TC1}} &= \frac{1}{2}\sum_{k=1}^{r}((p_k^+)^2 - (p_k^-)^2) - \frac{1}{2}(p_{r+1}^-)^2 \\ &\quad +2\sum_{k=1}^{r-1} \mathrm{e}^{(q_{k+1}^+ - q_k^+)/\sqrt{2}} \cos\frac{q_{k+1}^- - q_k^-}{\sqrt{2}} \\ &\quad +2\mathrm{e}^{-q_r^+/\sqrt{2}} \cos\left(q_{r+1}^- - \frac{q_r^-}{\sqrt{2}}\right), \end{aligned} \qquad (27)$$

$$\omega_{\mathbb{R}} = \sum_{k=1}^{r} \mathrm{d}p_k^+ \wedge \mathrm{d}q_k^+ - \sum_{k=1}^{r+1} \mathrm{d}p_k^- \wedge \mathrm{d}q_k^-. \qquad (28)$$

Here $q_k^{\pm}$ and $p_k^{\pm}$ are related to the initial q_k, p_k by:

$$p_k^{\pm} = \frac{1}{\sqrt{2}}(p_k \mp p_{\bar{k}}), \qquad q_k^{\pm} = \frac{1}{\sqrt{2}}(q_k \mp q_{\bar{k}}), \qquad (29)$$

for $k = 1, \ldots, r$ and

$$p_{r+1}^- = p_{r+1}, \qquad q_{r+1}^- = q_{r+1}, \qquad p_{r+1}^+ = q_{r+1}^+ = 0.$$

ii) for $n = 2r$:

$$\begin{aligned} H_{\mathrm{TC2}} &= \frac{1}{2}\sum_{k=1}^{r}((p_k^+)^2 - (p_k^-)^2) + \mathrm{e}^{-\sqrt{2}q_r^+} \\ &\quad +2\sum_{k=1}^{r-1} \mathrm{e}^{(q_{k+1}^+ - q_k^+)/\sqrt{2}} \cos\frac{q_{k+1}^- - q_k^-}{\sqrt{2}}, \end{aligned} \qquad (30)$$

$$\omega_{\mathbb{R}} = \sum_{k=1}^{r} \mathrm{d}p_k^+ \wedge \mathrm{d}q_k^+ - \sum_{k=1}^{r} \mathrm{d}p_k^- \wedge \mathrm{d}q_k^- \qquad (31)$$

the canonical coordinates $q_k^{\pm}$ and $p_k^{\pm}$ are as in equation (29).

We can easily obtain the solutions for each of the models (27) and (30) from the solutions of the CTC model (see *e.g.* [2] and the references therein) by just imposing the corresponding reductions on the initial parameters.

These models are generalizations of the well known Toda chain models associated to the classical Lie algebras [12]; indeed if we put $q_k^- \equiv 0$ and $p_k^- \equiv 0$ we find that (27) goes into the $\mathbf{B}_r$ TC while (30) provides the $\mathbf{C}_r$ TC.

5 Discussion

With each involutive automorphism $\tilde{\mathcal{C}}^2 = \mathbb{1}$ of the complexified phase space we were able to associate to a given Hamiltonian system $\mathcal{H}$ a new Hamiltonian system $\mathcal{H}_{\mathbb{R}}$. If the initial system $\mathcal{H}$ is integrable, so will be its real Hamiltonian form $\mathcal{H}_{\mathbb{R}}$. Obviously this method could be naturally extended to other types of Poisson brackets, *e.g.* the ones related with the structure constants of simple Lie algebras.

Our method provides an effective tool to derive new integrable Hamiltonian systems from a given one.

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